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LETTER TO THE EDITOR

On the trace anomaly of the energy–momentum tensor from the one-loop effective Lagrangian in QED

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Abstract. Using the results on one-loop corrections to the effective Lagrangian in QED for constant prescribed electromagnetic fields, we demonstrate a new way to derive the trace anomaly in spinor and scalar QED.

The past few years have witnessed an enormous interest in anomalies—mostly chiral anomalies—in various field theories. The methods for evaluating them have become increasingly sophisticated, from Schwinger's treatment in his 1951 paper [1] to Zumino's differential geometric approach [2] or Fujikawa's path integral treatment of the fermionic measure under chiral transformation [3], to mention just a few [4]. We will not be interested in this subject in the present letter, but want to concentrate on the equally important trace anomaly which was discussed years ago, e.g. in [5], with the result that in spinor QED:

$$-\langle T_\mu{}^\mu(x)\rangle = m\langle\bar{\psi}(x)\psi(x)\rangle + \frac{e^2}{24\pi^2} F_{\mu\nu}(x)F^{\mu\nu}(x). \qquad (1)$$

We want to explicitly compute $\langle T_\mu{}^\mu\rangle$ in terms of external electromagnetic fields and thus identify $\langle\bar{\psi}\psi\rangle$ as well as the anomaly which is obtained by $\lim_{m\to 0}\langle T_\mu{}^\mu\rangle$. The calculation is performed using knowledge of the heat kernel for the Dirac field in an external constant electromagnetic field. Among various approaches given, e.g. in [6], we are going to present still another derivation of the heat kernel, which is based on a WKB approximation.

The Lagrangian we are dealing with is given by ($\sigma_{\mu\nu} = \frac{1}{2}\mathrm{i}[\gamma_\mu, \gamma_\nu]$)

$$\mathcal{L} = \tfrac{1}{4}\dot{\eta}_\mu\dot{\eta}^\mu - \mathrm{i}e\dot{\eta}^\mu A_\mu(\eta) - \tfrac{1}{2}e\sigma_{\mu\nu}F^{\mu\nu}. \qquad (2)$$

Then the transition amplitude (in Euclidean time t) between events $\eta(0) = y$ and $\eta(t) = x$ can be written in terms of a Feynman path integral as

$$\langle x, t|y, 0\rangle = K(x, t; y, 0) = \int [\mathrm{d}\eta(\tau)] \exp\{-S[\eta(\tau)]\} \qquad (3)$$

where the classical action is simply

$$S[\eta(\tau)] = \int_0^t \mathrm{d}\tau\, \mathcal{L}(\eta, \dot{\eta}; \tau).$$

For constant fields, the path integral can be determined directly or by calculating the relevant Seeley coefficients, or—even more simply—by a classical WKB approximation which is exact for the constant-field case:

$$K(x, t; y, 0) = \frac{D^{1/2}(x, y; t)}{(4\pi t)^2} \exp[-S(x, y; t)] \tag{4}$$

with the Van Vleck determinant

$$D(x, y; t) = \left| \det\left(-2t \frac{\partial^2 S}{\partial x^\mu \, \partial y^\nu} \right) \right| \tag{5}$$

and the classical action:

$$S = -\mathrm{i}e \int_y^x \mathrm{d}\xi^\mu A_\mu(\xi) + \tfrac{1}{4}(x-y)^\mu e F_\mu{}^\rho (\cot eFt)_{\rho\nu}(x-y)^\nu - \tfrac{1}{2} e\sigma_{\mu\nu} F^{\mu\nu} t \tag{6}$$

where $\xi^\mu(\tau)$ denotes a straight line between x and y.

The Van Vleck determinant then yields

$$D^{1/2} = \exp\left(-\tfrac{1}{2} \operatorname{Tr} \ln \frac{\sin(eFt)}{eFt} \right).$$

The final result reproduces Schwinger's proper-time calculation:

$$K(x, t; y, 0) = \exp\left(\mathrm{i}e \int_y^x \mathrm{d}\xi^\mu A_\mu(\xi) \right) \frac{1}{(4\pi t)^2}$$
$$\times \exp\left(-\tfrac{1}{2} \operatorname{Tr} \ln \frac{\sin(eFt)}{eFt} \right) \exp[\tfrac{1}{2} e\sigma F t - \tfrac{1}{4}(x-y)^T eF(\cot eFt)(x-y)] \tag{7}$$

where the entire gauge dependence in the first factor has been isolated. The diagonal part, $x = y$, of (7) can be used to compute the effective action in QED:

$$\Gamma^{(1)} = \int \mathrm{d}^4x \, \mathscr{L}^{(1)} = \tfrac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{t} \exp(-m^2 t) \operatorname{Tr} \int \mathrm{d}^4x \, K(x, t; x, 0) + \text{constant}$$

which, when renormalised, leads to the effective Lagrangian [6] (for a constant magnetic field in the z direction):

$$\mathscr{L}^{(1)}[B] = -\frac{1}{32\pi^2} \{[2m^4 - 4m^2(eB) + \tfrac{4}{3}(eB)^2][1 + \ln(m^2/2eB)]$$
$$+ 4m^2(eB) - 3m^4 - (4eB)^2 \zeta'(-1, m^2/2eB)\}. \tag{8}$$

Now there is a very close relation between the effective Lagrangian $\mathscr{L}^{(1)}$ and the trace of the energy-momentum tensor:

$$\langle T_\mu{}^\mu(x) \rangle = m \frac{\partial \mathscr{L}^{(1)}(x)}{\partial m}. \tag{9}$$

The result of the mass differentiation is

$$\langle T_\mu{}^\mu \rangle(B) = -\frac{m}{32\pi^2} \{[8m^3 - 8m(eB)][1 + \ln(m^2/2eB)] + (2/m)[2m^4 - 4m^2(eB) + \tfrac{4}{3}(eB)^2]$$
$$+ 8m(eB) - 12m^3 - (4eB)^2 \frac{\partial}{\partial m} \zeta'(-1, m^2/2eB)\}. \tag{10}$$

When the Hurwitz ζ function $\zeta(z, q)$ is differentiated,

$$\frac{\partial}{\partial q}\zeta'(z, q)_{z=-1}=\frac{\partial}{\partial z}\left(\frac{\partial}{\partial q}\zeta(z, q)\right)\bigg|_{z=-1}$$
$$=\ln\Gamma(q)-\tfrac{1}{2}\ln 2\pi+q-\tfrac{1}{2}$$

we end up with

$$\langle T_\mu{}^\mu\rangle(B)=-\frac{1}{12\pi^2}e^2B^2-\frac{m^4}{4\pi^2}\ln\left(\frac{m^2}{2eB}\right)+\frac{m^2}{4\pi^2}eB\ln\left(\frac{m^2}{2eB}\right)+\frac{m^4}{4\pi^2}$$
$$+\frac{eBm^2}{2\pi^2}\left[\ln\Gamma\left(\frac{m^2}{2eB}\right)-\tfrac{1}{2}\ln 2\pi\right]. \tag{11}$$

This is also the result of the calculation of the integral

$$\langle T_\mu{}^\mu\rangle(B)=\frac{eBm^2}{4\pi^2}\int_0^\infty\frac{dz}{z^2}\exp\left(-\frac{m^2}{eB}z\right)(z\coth z-1-\tfrac{1}{3}z^2). \tag{12}$$

Now observe that

$$\lim_{m\to 0}\langle T_\mu{}^\mu\rangle=-\frac{1}{12\pi^2}e^2B^2 \tag{13}$$

which, when written covariantly, yields $(B^2\to\frac{1}{2}F_{\mu\nu}F^{\mu\nu})$

$$\lim_{m\to 0}\langle T_\mu{}^\mu\rangle=-\frac{1}{24\pi^2}e^2F_{\mu\nu}F^{\mu\nu}=-\frac{2\alpha}{3\pi}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{14}$$

which is the desired result.

Going back to our original equation (1), we can also identify

$$\langle\bar\psi\psi\rangle(B)=\frac{m^3}{4\pi^2}\ln\frac{m^2}{2eB}-\frac{eBm}{4\pi^2}\ln\frac{m^2}{2eB}-\frac{m^3}{4\pi^2}-\frac{eBm}{2\pi^2}\left[\ln\Gamma\left(\frac{m^2}{2eB}\right)-\tfrac{1}{2}\ln 2\pi\right]. \tag{15}$$

If we change from a constant B field in the z direction to a constant E field ($B\to(1/\mathrm{i})E$), we obtain a real and imaginary part for $\langle T_\mu{}^\mu\rangle(E)$:

$$\mathrm{Re}\langle T_\mu{}^\mu\rangle(E)=\frac{1}{12\pi^2}e^2E^2-\frac{m^4}{4\pi^2}\ln\frac{m^2}{2eE}+\frac{m^2eE}{8\pi}+\frac{m^4}{4\pi^2}+\frac{eEm^2}{2\pi^2}\mathrm{Im}\ln\Gamma\left(\mathrm{i}\frac{m^2}{2eE}\right)$$

$$\mathrm{Im}\langle T_\mu{}^\mu\rangle(E)=-\frac{m^4}{8\pi}-\frac{m^2}{4\pi^2}eE\ln\frac{m^2}{2eE}-\frac{eEm^2}{2\pi^2}\left[\mathrm{Re}\ln\Gamma\left(\mathrm{i}\frac{m^2}{2eE}\right)-\tfrac{1}{2}\ln 2\pi\right]$$
$$=-\frac{m^4}{8\pi}+\frac{eEm^2}{4\pi^2}\ln\left(2\sinh\frac{\tau m^2}{2eE}\right).$$

As a consistency check with Schwinger's formula for pair production, we obtain

$$\mathrm{Im}\langle T_\mu{}^\mu\rangle(E)=m\frac{\partial}{\partial m}\mathrm{Im}\,\mathscr{L}^{(1)}(E)=-\frac{m^2eE}{4\pi^2}\sum_{M=1}^{\infty}\frac{1}{M}\exp\left(-\frac{\pi m^2}{eE}M\right). \tag{16}$$

Similar relations hold for scalar QED where, with the aid of

$$\mathscr{L}_0^{(1)}(B)=\frac{1}{64\pi^2}\left[[2m^4-\tfrac{2}{3}(eB)^2]\left(1+\ln\frac{m^2}{2eB}\right)-3m^4-(4eB)^2\zeta'\left(-1,\frac{m^2+eB}{2eB}\right)\right] \tag{17}$$

we find

$$\langle T_\mu{}^\mu\rangle(B) = 2\frac{\partial \mathscr{L}_0^{(1)}(B)}{\partial \ln m^2}$$

$$\langle T_\mu{}^\mu\rangle = \frac{m}{64\pi^2}\left\{-8m^3 - \frac{4}{3}\frac{(eB)^2}{m} + 8m^3\ln\frac{m^2}{2eB} - 16m(eB)\left[\ln\Gamma\left(\frac{m^2+eB}{2eB}\right) - \tfrac{1}{2}\ln 2\pi\right]\right\} \tag{18}$$

where the anomaly is contained in the $(eB)^2$ term.

Of course, we can also obtain the next-to-leading term in the radiatively corrected formula [7], for spin $\frac{1}{2}$:

$$\langle T_\mu{}^\mu\rangle_{m=0} = -\beta(\alpha)\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \beta(\alpha) = \frac{2}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 + \cdots \tag{19}$$

by incorporating our results for the two-loop calculation $\mathscr{L}^{(2)}$ extensively outlined in [6].

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